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From $E = mc^2$ to the Lorentz transformations via the law of addition of relativistic velocities.

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Abstract. In this paper we show how to get the Lorentz transformations from $E = mc^2$, the laws of conservation of energy and momentum, and the special relativity principle. To this end we first deduce the law of addition of relativistic velocities.

1. Introduction

This year is the centenary of the 1905 Einstein *Agnus Mirabilis* [1]. In that year he published five important papers for the future of the physics. One of them [1] was about the photoelectric effect, and for it Einstein won the 1921 Nobel prize in physics; two of them [1] were about statistic mechanics, and the others two on what was later known as the special theory of relativity. The first of those papers [2], *On the electrodynamic of Moving Bodies*, was published on June 30, and contains the basic theory of special relativity. In that paper he proved the relativistic formula of addition of velocities. This formula shows *"that the velocity of the light c cannot be altered by addition with a velocity less than the light"* (see [2], p.51), and therefore he solved the problem motivated by the fact *"that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body"* (see [2], p.38). Three months later he published: *"Does the inertia of a body depend upon its energy-content?"* [3]. In that paper he concluded that *"the mass of a body is a measure of its energy content"*. This led to the famous formula $E = mc^2$. In this paper we reverse in some sense the logic of the expository order of those papers. Our basic physical assumptions are:

(i) The mass-energy equivalence relation, (ii) the energy conservation law, (iii) the momentum conservation law, (iv) the special relativity principle. From these assumptions we show how can be deduced the relativistic formula of addition of velocities, and then the *Lorentz Transformations*(LT).

2. Addition of parallel and perpendicular velocities

To simplify our exposition we choose the *standard* coordinates (x, y, z, t) and (x', y', z', t') in two arbitrary *inertial frames*(IF) S and S', in the *standard configuration*, that is, the S' origin moves with velocity $\mathbf{v}$ along the x-axis of S, the x'-axis coincides with the x-axis, while the y- and y'-axes remain parallel, as do the z- and z'-axes; and all clocks are set to zero when the two origins meet (see [4], p. 43). We have assumed the *Special Relativity Principle*(SRP), and it is well known that from the SRP follows the homogeneity and isotropy of all the IF, but as it is noted in Ref. [4], p. 40, it is perhaps less well known that, conversely, the homogeneity and isotropy of all the IF imply the SRP. Now $y' = y$ and $z' = z$ follow by assuming that the transformation between IF is linear and the SRP holds. In fact, linearity follows from Newton's first law and temporal and spatial homogeneity (see Ref. [4], p. 43).

Our starting point will be the equivalence between mass and energy, that is, $E = mc^2$, where E is the energy content of one *particle*, c is the vacuum velocity of the light, and m is the *inertial mass* of the particle, which is given by $m_0\gamma_u$, where $\gamma_u = (1 - u^2/c^2)^{1/2}$ in an IF of reference in which the particle moves with velocity $\mathbf{u}$ ($u = |\mathbf{u}|$), and m_0 is the rest mass of the particle. The relativistic momentum is given by $\mathbf{p} = m\mathbf{u}$. Consider a massless box, which contains two particles, 1 and 2, with equal rest mass m_0, moving along a straight line with speeds **-u** and $\mathbf{u}$ respectively, in the IF

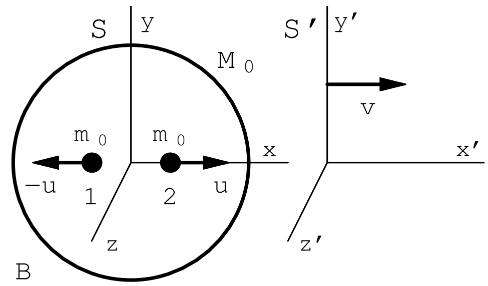

Figure 1. The rest mass M_0 of the *box-particle* B is the sum of the energy of the two particles 1 and 2. To find the addition of the parallel velocities $\mathbf{u}$ and $\mathbf{v}$, we calculate the total energy and the total momentum in S' by two ways: as one unique particle B and as two separated particles 1 and 2.

S in which the box is at rest, so S is the *zero-momentum frame*([4], p.117).

We will calculate now the total energy and the total momentum of the system of the two particles in the frame S' by two different ways: one by considering the system as a unique *box-particle* B, whose rest mass M_0 is a measure of the energy content of the box, and the other by considering the system as two separated particles.

In the first case the total energy of B in the frame S' is given by $E = M_0\gamma_v c^2$, and by use of the Einstein's assumption that M_0 is a measure of the energy content of the box B, $M_0 = 2m_0\gamma_u$, so $E = 2m_0\gamma_u\gamma_v c^2$. In the same way the momentum of B in S' is given by $\mathbf{p} = M_0\gamma_v\mathbf{v} = 2m_0\gamma_u\gamma_v\mathbf{v}$.

In the second case $E = m_0\gamma_{v_1}c^2 + m_0\gamma_{v_2}c^2$, and $\mathbf{p} = m_0\gamma_{v_1}\mathbf{v}_1 + m_0\gamma_{v_2}\mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are respectively the velocities of the particles 1 and 2 in the frame S'. Comparing the energy E and the momentum $\mathbf{p}$ calculated by the two ways we obtain:

$$2\gamma_u\gamma_v = \gamma_{v_1} + \gamma_{v_2}, \quad 2\gamma_u\gamma_v\mathbf{v} = \gamma_{v_1}\mathbf{v}_1 + \gamma_{v_2}\mathbf{v}_2 \tag{2.1}$$

We will find now the formula of addition of the velocities $\mathbf{u}$ and $\mathbf{v}$ for the cases in which they are parallel or perpendicular. In the following it is assumed that coordinate transformations between IF include all rigid motions in the 3-space, in particular rotations and space-reflections which are the transformations used in this paper.

2.1. Addition of parallel velocities

Consider that the particles 1 and 2 of the massless box move along the x-axis (see Fig. 1). This configuration is invariant under rotations about x-axis in S, and then from the SRP this configuration must be invariant under rotations about x'-axis in S' too. That follows from the fact that the rotations about x-axis commute with the transformation between the IF S and S', because in the *standard configuration* the x'-axis coincides with the x-axis. From that it follows that $\mathbf{v}_1$ and $\mathbf{v}_2$ have both the direction of the x'-axis and so

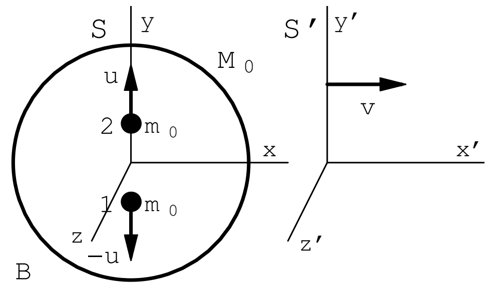

Figure 2. We use this disposition of the particles 1 and 2, to find the addition of the perpendicular velocities **u** and **v**.

we can denote them by v_1 and v_2 §. Moreover, if we take $c = 1$ and use as new variables the rapidities χ_1, χ_2, α_u, and α_v defined by $\tanh\chi_1 = v_1$, $\tanh\chi_2 = v_2$, $\tanh\alpha_u = u$, and $\tanh\alpha_v = v$, the Eqs. (2.1) become:

$$2\cosh\alpha_u\cosh\alpha_v = \cosh\chi_1 + \cosh\chi_2 = 2\cosh\frac{\chi_1+\chi_2}{2}\cosh\frac{\chi_1-\chi_2}{2}, \quad (2.2)$$

$$2\cosh\alpha_u\cosh\alpha_v v = \sinh\chi_1 + \sinh\chi_2 = 2\sinh\frac{\chi_1+\chi_2}{2}\cosh\frac{\chi_1-\chi_2}{2}. \quad (2.3)$$

Dividing member by member Eq. 2.3 by Eq. 2.2, we get $v = \tanh\frac{\chi_1+\chi_2}{2}$, and then $\frac{\chi_1+\chi_2}{2} = \alpha_v$.

Comparing with Eq. 2.2 we get $\cosh\frac{\chi_1-\chi_2}{2} = \cosh\alpha_u$, and then $\frac{\chi_1-\chi_2}{2} = \alpha_u$. Therefore, $\chi_1 = \alpha_v + \alpha_u$ and $\chi_2 = \alpha_v - \alpha_u$.

Finally we obtain the following expression for the velocity of particle 1:

$$v_1 = \tanh\chi_1 = \frac{\tanh\alpha_v + \tanh\alpha_u}{1 + \tanh\alpha_v\tanh\alpha_u} = \frac{u+v}{1+uv}, \quad (2.4)$$

and, similarly, for particle 2:

$$v_2 = \frac{u-v}{1-uv} \quad (2.5)$$

These last equations are precisely the formulas of addition of velocities when **u** and **v** are parallel.

2.2. Addition of perpendicular velocities

We shall now study the case of **u** being perpendicular to **v**. In this case we consider that the particles 1 and 2 of the massless box move along the y-axis (see Fig. 2). In this configuration both particles have coordinate $z = 0$ for any time t in S. Then $z' = 0$ (because $z' = z$) for both particles and for any time t' in S'. This implies that $\mathbf{v}_1$ and $\mathbf{v}_2$ must be in the plane $x'y'$. Since the configuration is invariant under the reflection

§ Other way to see that both $\mathbf{v}_1$ and $\mathbf{v}_2$ are along the x'-axis is to observe that in S $y = z = 0$ for any time t for both particles, and then, because $y' = y$ and $z' = z$, $y' = z' = 0$ for any time t' for both particles in S', and then $\mathbf{v}_1$ and $\mathbf{v}_2$ must be along the x'-axis.

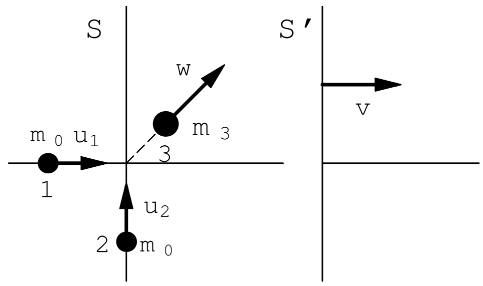

Figure 3. We use the conservation of energy and momentum in the inelastic collision of particles 1 and 2, to find the general formulas for the transformation of velocities.

$y \leftrightarrow -y$ in S, and this reflection commute with the transformation between the IF S and S', then the configuration in S' must be invariant under the reflection $y' \leftrightarrow -y'$. This implies that $\mathbf{v}_1$ and $\mathbf{v}_2$ must be in the plane $x'y'$. On the other hand, the invariance under $y' \leftrightarrow -y'$ implies that $v_1 = v_2$ and the angle between $\mathbf{v}_1$ and the x'-axis, θ, is the opposite to the angle between $\mathbf{v}_2$ and the x'-axis. Then $\mathbf{v}_1 = v_1(\cos\theta, -\sin\theta)$, $\mathbf{v}_2 = v_1(\cos\theta, \sin\theta)$. Now from Eqs. (2.1) we get:

$$\mathbf{v} = (v, 0) = \frac{\gamma_{v_1}\mathbf{v}_1 + \gamma_{v_2}\mathbf{v}_2}{\gamma_{v_1} + \gamma_{v_2}}. \tag{2.6}$$

Therefore $v = v_1 \cos\theta$, and from the first equation of (2.1) it follows that $\cosh\alpha_u \cosh\alpha_v = \cosh\chi_1$. This last formula in velocity parameters is $\gamma_u\gamma_v = (1 - (\frac{v}{\cos\theta})^2)^{-1/2}$. Using the expression of $\cos\theta$ obtained from this formula we finally obtain:

$$\mathbf{v}_1 = v_1(\cos\theta, -\sin\theta) = (v, -u\sqrt{1 - v^2}), \tag{2.7}$$

and

$$\mathbf{v}_2 = (v, u\sqrt{1 - v^2}). \tag{2.8}$$

These equations correspond to the formulas of addition of the perpendicular velocities $\mathbf{u}$ and $\mathbf{v}$.

3. Transformation of the velocities

We will now get the general formulas for the transformation of the components of the velocities in a change of reference frame. To this end, consider the collision of two particles 1 and 2, with the same rest mass m_0, and with orthogonal velocities $\mathbf{u}_1$ and $\mathbf{u}_2$. Let m_3 be the rest mass of the particle 3 resulting from the collision, and $\mathbf{w}$ its velocity in an IF S (see Fig. 3). We also denote by $\mathbf{u}'_1$, $\mathbf{u}'_2$ and $\mathbf{w}'$ the corresponding velocities in the frame S' that moves with velocity $\mathbf{v}$ along the direction of $\mathbf{u}_1$. Using the conservation of the energy and the momentum we get:

$$E = m_0\gamma_{u_1}c^2 + m_0\gamma_{u_2}c^2 = m_3\gamma_w c^2, \quad E' = m_0\gamma_{u'_1}c^2 + m_0\gamma_{u'_2}c^2 = m_3\gamma_{w'}c^2 \tag{3.1}$$

$$\mathbf{p} = m_0\gamma_{u_1}\mathbf{u}_1 + m_0\gamma_{u_2}\mathbf{u}_2 = m_3\gamma_w\mathbf{w}, \quad \mathbf{p}' = m_0\gamma_{u'_1}\mathbf{u}'_1 + m_0\gamma_{u'_2}\mathbf{u}'_2 = m_3\gamma_{w'}\mathbf{w}' \tag{3.2}$$

From this we get: $\mathbf{w} = (w_1, w_2) = (\frac{u_1\gamma_{u_1}}{\gamma_{u_1}+\gamma_{u_2}}, \frac{u_2\gamma_{u_2}}{\gamma_{u_1}+\gamma_{u_2}})$, and a similar expression follows for $\mathbf{w}'$.

From the results of Sections 2.1 and 2.2, it follows that $\mathbf{u}_1' = (\frac{u_1-v}{1-vu_1}, 0)$, and $\mathbf{u}_2' = (v, u_2\sqrt{1-v^2})$. Finally, an easy calculation using the well-known relations $\gamma_{u_1'} = (1-vu_1)\gamma_{u_1}\gamma_v$ and $\gamma_{u_2'} = \gamma_{u_2}\gamma_v$ (see [4], p. 69) gives the expected formulas:

$$w_1' = \frac{(u_1 - v)\gamma_{u_1} + v\gamma_{u_2}}{(1 - vu_1)\gamma_{u_1} + \gamma_{u_2}} = \frac{w_1 - v}{1 - vw_1}, \tag{3.3}$$

$$w_2' = \frac{u_2\sqrt{1-v^2}\gamma_{u_2}}{(1 - vu_1)\gamma_{u_1} + \gamma_{u_2}} = \frac{w_2\sqrt{1-v^2}}{1 - vw_1}. \tag{3.4}$$

Observe that, equivalently, the above calculation can be done considering the movement of the center of mass of the system.

4. The Lorentz Transformations

The Lorentz Transformations can be obtained from the transformation of velocities if we assume the SRP. To see this, choose as above the *standard* coordinates (x, y, z, t) and (x', y', z', t') in two arbitrary IF S and S', in the *standard configuration*. Consider a particle moving with uniform velocity from the spacial origin for time zero to the point $(x, y, 0)$ for time t in the frame S, so its velocity in S is given by: $\mathbf{w} = (w_1, w_2, 0) = (x/t, y/t, 0)$. Let (x', y', z', t') the the associated coordinates to $(x, y, 0, t)$ in S'. As it has been noted in the first paragraph of Section 2, from the SRP, it follows that $y' = y$ and $z' = z$. The velocity in S' is given by $\mathbf{w}' = (w_1', w_2', 0) = (x'/t', y'/t', 0)$. If we now use the formulas (3.3) and (3.4) of transformation of velocities we get:

$$x'/t' = \frac{x/t - v}{1 - vx/t} \tag{4.1}$$

$$y'/t' = \frac{y/t\sqrt{1-v^2}}{1 - vx/t} \tag{4.2}$$

Now, since $y' = y$ and $z' = z = 0$, we get:

$$x' = \frac{x - vt}{\sqrt{1-v^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - vx}{\sqrt{1-v^2}} \tag{4.3}$$

The homogeneity and isotropy of all the IF implies that these transformations are valid for the coordinates (x, y, z, t) associated to any *event* in S. Thus we have found *the standard Lorentz transformation equations.*

5. Conclusions

We have proved that the law of addition of velocities can be obtained from $E = mc^2$, the conservation of energy and momentum, and the SRP. Moreover, the Lorentz transformations are obtained immediately from the law of addition of velocities and the SRP.

Thus, taking $E = mc^2$ and the conservation of energy and momentum, together with the SRP as the starting principles of special relativity, is a possible option. In fact, the relation between energy and inertial mass was pointed out in some particular cases 24 years before Einstein's fundamental paper ‖. Moreover, $E = mc^2$ can be considered as an extension of the First Principle of the Thermodynamic, and as such, a good Principle for the foundation of any theory. However we have to accept that although the derivation of the LT in this paper is logically interesting, it presupposes ad hoc the strongly counterintuitive assumptions of the mass-energy and mass-velocity relations. Therefore, it makes little sense as a means to conceive the LT (in other words, the premises from which the LT follows are conceptually equally complicated with their consequence, namely the LT). Consequently, it is didactically of limited interest.

We also note that if we take the terms in u^2 and v^2 in the power-series expansion of the total energy, $E = 2m_0\gamma_u\gamma_v c^2$, of the compound particle B, we get the Newtonian total energy of B. Then, the same reasoning of this paper leads, as it is expected, to the Galileo transformations via the Galileo addition of velocities ¶.

Acknowledgments

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‖ As Thomson noted in 1881, this association is already implicit in Maxwell's theory [5]. It must also be mentioned the works of Poincaré, Abraham and Lorentz [6] about the electron mass arising from its electromagnetic energy, and the work of Hasenöhrl [7] about how the mass of a cavity increases when it is filled with radiation.

¶ With the configuration of the particles given in Section 2, the total energy of the *box-particle* B in the frame S' is given by the kinetic energy of a particle of mass $2m$ moving with the velocity v of the mass centre (origin of the frame S), plus the kinetic energy of the two particles 1 and 2 with respect to the mass centre. On the other hand, the kinetic energy of particles 1 and 2 in the frame S' is given by: $\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$ where v_1 and v_2 are the velocities of the particles 1 and 2 respectively, in the frame S'. Therefore, comparing the energy calculated by these two ways we get: $\frac{1}{2}2mv^2 + 2\frac{1}{2}mu^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$, and for the total momentum we have $2mv = mv_1 + mv_2$. From these equations we obtain: $v_1 = u+v$ and $v_2 = u - v$. A similar calculus to that of Sections 2.1 and 2.2 give the corresponding Galilean formulas for the composition of velocities. Finally reasoning as in Section V we get the Galilean coordinate transformations.

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